

## Lecture 19. The Finite Element Method in Elastic Body Mechanics

1. The essence of finite element method (FEM) and calculation of finite element's potential energy.
2. Internal strain energy of elastic body in FEM-approximation.
3. The equations of FEM and method's interpretation by the use of nodal model of elastic body.
4. The example of calculations.

1

The FEM is the realization of principle of virtual work for computation of stressed-strained state (SSS) of deformed systems [30: sect. 2.2]. To ground FEM for elastic systems one might apply the principle of complete potential energy minimization. Further this method is under consideration for the case of fixed elastic body [30, 16].

In FEM, the elastic body state is conditioned by the finite number of degrees of freedom (DOF). Accordingly to this method, the displacements of certain body's points, named nodes, are taken as the DOFs. In the widespread method's realization the nodes are the vertices of polyhedrons, from which the elastic body consists of, as we imagine. By the method, there are specified elastic properties for each polyhedron, more distinctly, there are established approximate relationships between strains, stresses, and, finally, internal strain energy, on one side, and the displacements of polyhedron's vertices, on the other. The error of the internal strain energy evaluation by means of this relationships must be infinitesimal at the polyhedron's size minimization. If this relationships take place, the polyhedrons are named *finite elements*. **FEM is based on 1) the representation of elastic body strain energy by the displacements of nodes, and 2) making up linear equations for the displacements, securing the minimum for the complete potential energy of elastic body. Solving the equations, we determine actual displacements and, finally, the SSS.**

We shall be bounded by the case of plane fixed elastic body with thickness  $\delta$  at plane stressed state. We consider body displacements in the coordinate system (CS)  $O'xy$ , linked to the earth. The displacements are denoted, as usual,  $u$  and  $v$ . We'll specify the nodal mesh to make coincidence for points of support and load application with nodes. We suppose, that every bearing eliminates one of the DOFs  $x$ ,  $y$  or both of them simultaneously. In other words, the displacements are restricted as follows:

$$u = 0 \quad (19.1)$$

or

$$v = 0 \quad (19.2)$$

or

$$u = 0 \text{ and } v = 0. \quad (19.3)$$

We'll consider the case of triangle finite elements with **linear law** for displacement variation over each element spread. This assumption causes that everyone of triangles keeps its triangle shape after deformation, and, therefore, there is continuity of the material under deformation. Having represented the location of non-deformed element by nodal coordinates, we'll express the internal strain energy of element through nodal displacements (Fig. 19.1).

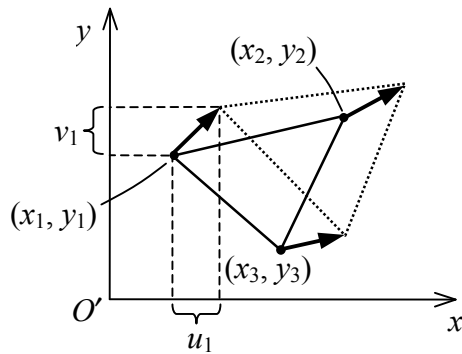


Fig. 19.1 Finite element before deformation (solid contour) and after deformation (dash contour)

Introduce

$$\mathbf{t}^T = (u_1, u_2, u_3, v_1, v_2, v_3) \quad (19.4)$$

— vector of element's vertices displacements (index indicates the number of vertex);

$$\mathbf{e}^T = (\epsilon_x, \epsilon_y, \gamma_{xy})$$

— strain vector over element boundary;

$$\mathbf{S}^T = (\sigma_x, \sigma_y, \tau_{xy})$$

— stress vector over element boundary.

Strain vector in element's boundary doesn't depend on coordinates due to displacements linearity on coordinates. Denote by  $\mathbf{A}$  the strain-displacement matrix, that is such matrix, that  $\mathbf{e} = \mathbf{A}\mathbf{t}$ ; by  $\mathbf{C}$  the Houke's law matrix, that is such matrix, that  $\mathbf{S} = \mathbf{C}\mathbf{e}$ . Matrix  $\mathbf{C}$  has the form:

$$\mathbf{C} = \frac{E}{1-\mu^2} \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{pmatrix}. \quad (19.5)$$

Obviously, it is symmetrical.

For internal strain energy of element of volume  $V$  we have:

$$U_1 = \frac{V}{2} \mathbf{S}^T \mathbf{e} = \frac{V}{2} (\mathbf{C}\mathbf{e})^T \mathbf{e} = \frac{V}{2} \mathbf{e}^T \mathbf{C}^T \mathbf{e} = \frac{V}{2} (\mathbf{A}\mathbf{t})^T \mathbf{C} (\mathbf{A}\mathbf{t}), \quad (19.6)$$

or finally:

$$U_1 = \frac{V}{2} \mathbf{t}^T \mathbf{A}^T \mathbf{C} \mathbf{A} \mathbf{t} = \frac{1}{2} \mathbf{t}^T \mathbf{R} \mathbf{t}, \quad (19.7)$$

where introduced finite element stiffness matrix:

$$\mathbf{R} = V\mathbf{A}^T\mathbf{C}\mathbf{A}. \quad (19.8)$$

We shall obtain the strain-displacement matrix  $\mathbf{A}$ . Write out linear relationship for displacements in finite element's boundary:

$$\begin{aligned} u &= a_1x + b_1y + c_1; \\ v &= a_2x + b_2y + c_2. \end{aligned} \quad (19.9)$$

We have:

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_1; \quad \varepsilon_y = \frac{\partial v}{\partial y} = b_2; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = b_1 + a_2. \quad (19.10)$$

We express coefficients in the right-hand sides through triangle's vertices displacements, solving two systems of equations:

$$\begin{cases} a_1x_1 + b_1y_1 + c_1 = u_1; \\ a_1x_2 + b_1y_2 + c_1 = u_2; \\ a_1x_3 + b_1y_3 + c_1 = u_3; \end{cases} \quad \begin{cases} a_2x_1 + b_2y_1 + c_2 = v_1; \\ a_2x_2 + b_2y_2 + c_2 = v_2; \\ a_2x_3 + b_2y_3 + c_2 = v_3. \end{cases} \quad (19.11)$$

The Cramer's rule gives:

$$\begin{aligned} a_1 &= \Delta^{-1}[u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2)]; \\ b_1 &= \Delta^{-1}[u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1)]; \\ a_2 &= \Delta^{-1}[v_1(y_2 - y_3) + v_2(y_3 - y_1) + v_3(y_1 - y_2)]; \\ b_2 &= \Delta^{-1}[v_1(x_3 - x_2) + v_2(x_1 - x_3) + v_3(x_2 - x_1)], \end{aligned} \quad (19.12)$$

where:

$$\Delta = \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

It is marked in the book [16], that the determinant is equal numerically to double triangle's square. We might add it is positive at vertices' enumeration contra hour hand move.

After substitution into (19.10) corresponding to (19.12) we get the wanted matrix:

$$\mathbf{A} = \Delta^{-1} \begin{pmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{pmatrix}. \quad (19.13)$$

It completes the investigations of triangle finite element properties: the internal strain energy has the form (19.7), defined by stiffness matrix (19.8) with

substitution (19.13).

## 2

Next we solve the problem of the representation for total strain energy of elastic body through the body's DOFs. Introduce vector  $\mathbf{t}$  of DOFs of fixed body, which is composed by selection of independent displacements out of full set of nodal displacements

$$(u_1, v_1, u_2, v_2, \dots, u_{n_1}, v_{n_1}), \quad (19.14)$$

where enumeration is used for all  $n_1$  nodes of the mesh. The vector  $\mathbf{t}$  dimensionality is equal to the number of DOFs  $n < 2n_1$ . For example, fig. 19.2 shows two-elements model of plane triangle wedge. Here  $n_1 = 4$ ,  $n = 3$ , the result of selection of independent displacements out of set (19.14) is as follows:

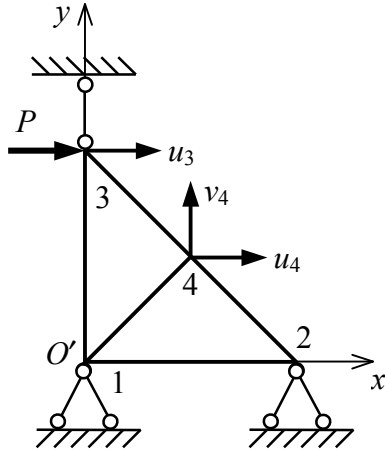


Fig. 19.2

$$\mathbf{t}^T = (u_3, u_4, v_4). \quad (19.15)$$

Let the model has  $m$  finite elements. The nodal displacements of  $k$ -th element we specify by the vector  $\mathbf{t}_k$  of the form (19.4). Its components are defined by body's DOFs, i.e. for any element holds:

$$\mathbf{t}_k = \mathbf{I}_k \mathbf{t}. \quad (19.16)$$

Here  $\mathbf{I}_k$  is the displacements' conversion matrix for  $k$ -th element. Every row of this matrix has only one component which may be nonzero. This component is equal to unit and defines the coordinate of vector  $\mathbf{t}$  specifying corresponding coordinate of vector  $\mathbf{t}_k$ . For instance, the model at fig. 19.2 is composed with two elements. Vectors of elements' nodal displacements are as follows (vertices' enumeration contra hour hand move):

$$\mathbf{t}_1^T = (u_1, u_2, u_4, v_1, v_2, v_4) = (0, 0, u_4, 0, 0, v_4);$$

$$\mathbf{t}_2^T = (u_1, u_4, u_3, v_1, v_4, v_3) = (0, u_4, u_3, 0, v_4, 0).$$

Matrix  $\mathbf{I}_1$ , for example, takes the form:

$$\mathbf{I}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We recommend to reader: check the correctness of the equality (19.16) for the first element and obtain the displacements' conversion matrix for the second.

Using the vectors of element's nodal displacements, we establish the internal strain energy of the whole body. For a given nodal mesh and finite elements ensemble we can evaluate the stiffness matrix (19.8) for any element. Denote stiffness matrix of  $k$ -th element  $\mathbf{R}_k$  ( $k = \overline{1, m}$ ). The strain energy of the whole body we obtain by summarization of strain energies for any element. The later we evaluate by (19.7):

$$U_1 = \sum_{k=1}^m U_{1k} = \frac{1}{2} \sum_{k=1}^m \mathbf{t}_k^T \mathbf{R}_k \mathbf{t}_k .$$

Using substitution (19.16), we represent initial strain energy as the function of DOF-vector:

$$U_1 = \frac{1}{2} \sum_{k=1}^m \mathbf{t}^T \mathbf{I}_k^T \mathbf{R}_k \mathbf{I}_k \mathbf{t} = \frac{1}{2} \mathbf{t}^T \left( \sum_{k=1}^m \mathbf{I}_k^T \mathbf{R}_k \mathbf{I}_k \right) \mathbf{t} .$$

Introducing the stiffness matrix of nodal system  $\mathbf{R}$ , we finally obtain:

$$U_1 = \frac{1}{2} \mathbf{t}^T \mathbf{R} \mathbf{t}; \quad (19.17)$$

$$\mathbf{R} \equiv \sum_{k=1}^m \mathbf{I}_k^T \mathbf{R}_k \mathbf{I}_k . \quad (19.18)$$

### 3

Compose the row of loads, applied to all body's nodes, as follows:

$$(P_{x1}, P_{y1}, P_{x2}, P_{y2}, \dots, P_{xn_1}, P_{yn_1}) . \quad (19.19)$$

Here we suppose for the nodes being unloaded:

$$P_{xi} = P_{yi} = 0. \quad (19.20)$$

Out of this row we'll compose the load vector  $\mathbf{P}$  by selection of components due to the same rule that let to compose before the DOF-vector  $\mathbf{t}$  out of the row (19.14). For example, in the case of the model on fig. 19.2 we have:

$$\mathbf{P}^T = (P, 0, 0) .$$

The external work due to the displacements of given node has the form:

$$A_i = P_{xi} u_i + P_{yi} v_i .$$

The complete external work we obtain by summarization of these works and write as follows:  $A = \mathbf{P}^T \mathbf{t}$ . Thus, the body's potential energy in external field takes the form:

$$U_2 = -\mathbf{P}^T \mathbf{t}. \quad (19.21)$$

For complete potential energy of elastic body we have the expression:

$$U = \frac{1}{2} \mathbf{t}^T \mathbf{R} \mathbf{t} - \mathbf{P}^T \mathbf{t}. \quad (19.22)$$

The displacements securing stable equilibrium of the body in external field is obtained by minimization of the function (19.22). To find the minimum we introduce the partial derivatives' vector

$$\left( \frac{\partial U}{\partial \mathbf{t}} \right)^T \equiv \left( \frac{\partial U}{\partial t_1}, \dots, \frac{\partial U}{\partial t_n} \right)$$

and equate it to zero. For function (19.19) it is easy to obtain:

$$\frac{\partial U}{\partial \mathbf{t}} = \mathbf{R} \mathbf{t} - \mathbf{P}.$$

Here we use the formula of quadratic form differentiation (19.1) taking into account the matrix  $\mathbf{R}$  is symmetrical. For the DOF-vector we get the equation:

$$\mathbf{R} \mathbf{t} - \mathbf{P} = 0. \quad (19.23)$$

Let's remember, that DOF-vector was formed by selection of DOFs from the full set of nodal displacements (19.14). It might be proved, that for such vector  $\mathbf{t}$  the stiffness matrix is not singular (see the supplement to the lecture). That's why there exists solution of the equation (19.23) at any loading, and it can be obtained by conversion of stiffness matrix as follows:

$$\mathbf{t} = \mathbf{R}^{-1} \mathbf{P}. \quad (19.24)$$

The physical essence of stiffness matrix can be stated, if we note that for any possible displacements it determines load vector

$$\mathbf{P} = \mathbf{R} \mathbf{t} = \sum_{i=1}^n \mathbf{R}_{*i} t_i, \quad (19.25)$$

causing these displacements. Here  $\mathbf{R}_{*i}$  is  $i$ -th column of stiffness matrix. From the later formula we see, that the state of unit displacement  $t_i = 1$  is invoked by the external force vector  $\mathbf{R}_{*i}$ . In other words, every column of stiffness matrix is the load vector, causing the unit displacement of a single node.

Obvious interpretation of FEM is by replacing of solid system to discreet system of material points placed at the nodes, with the same ensemble of loads. Equivalence conditions of the given body and replacing system are nodal displacements coincidence and bearing reactions coincidence. In replacing system the every particle is subjected to forces of other particles interaction, ground interaction forces (bearing reactions) and loading. Denote  $\mathbf{N}$  the vector of internal forces, applied to nodes in DOF directions. Bearing reaction along

permissible displacement is zero, therefore we have the equilibrium equations system for a system of nodes:  $\mathbf{N} + \mathbf{P} = 0$ . That yields:

$$\mathbf{N} = -\mathbf{P} = -\mathbf{R}\mathbf{t}.$$

Later relationship means, that stiffness matrix gives the rule to determine internal forces corresponding to nodal displacements.

Let's notice in conclusion the offered realization of FEM is the simplest, and that makes its advantage. For the partitioning of the body up to 2÷3 elements, it is easy to determine nodal displacements even by manual calculation by the use of the formula (19.24). However, this model doesn't have any means to determine bearing reactions and it doesn't support the case of "oblique" bearing (when instead of conditions (19.1)—(19.3) the virtual nodal displacement is given by condition  $u = kv$ ,  $k \neq 0$ ). In the supplement to lecture it is offered more complex realization of FEM, given in [31] and having got rid of mentioned limitations.

#### 4

Let rectangular wedge has fixed lower bound and is loaded by distributed loading (Fig. 19.3, *a*). Deformed state of finite element model is related to number of elements (Fig. 19.4). Let's carry out SSS-calculation by means of single-element model on Fig. 19.3, *b*. In this model the distributed load has been replaced by concentrated load  $P$  that makes the same work as the distributed one. It is easy to notice, that:

$$P = \frac{qa}{2}.$$

In this example the DOF-vector is as follows:

$$\mathbf{t}^T = (u_3, v_3). \quad (19.26)$$

Corresponding stiffness matrix  $\mathbf{R}$  is of dimensionality 2. This matrix is computed by the formula (19.18) through the element stiffness matrix  $\mathbf{R}_1$  and the conversion matrix  $\mathbf{I}_1$ . Beforehand there should be calculated the strain-displacement matrix  $\mathbf{A}_1$  of the form (19.13) for the calculation matrix  $\mathbf{R}_1$  through the formula (19.8). Let's rewrite the last formula, indicating the number of finite element:

$$\mathbf{R}_1 = V_1 \mathbf{A}_1^T \mathbf{C} \mathbf{A}_1.$$

For manual calculations this formula is not convenient because matrix  $\mathbf{R}_1$  is of high enough dimensionality (equal to 6). It's more convenient to obtain the product  $\mathbf{A}_1 \mathbf{I}_1$ , and then to calculate the stiffness matrix as follows:

$$\mathbf{R} = V_1 (\mathbf{A}_1 \mathbf{I}_1)^T \mathbf{C} (\mathbf{A}_1 \mathbf{I}_1).$$

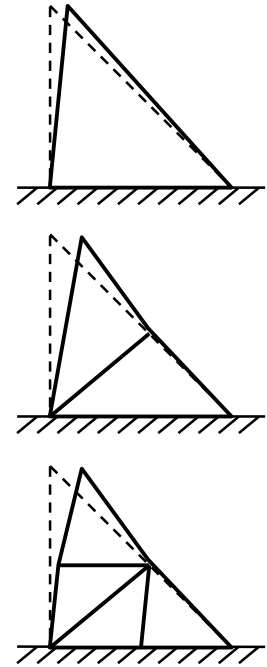
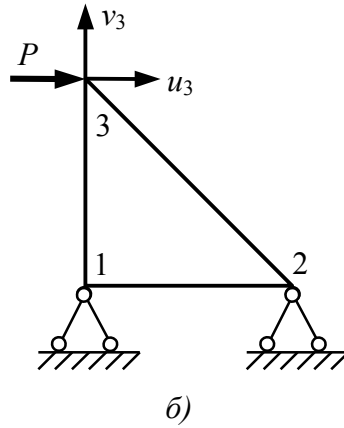
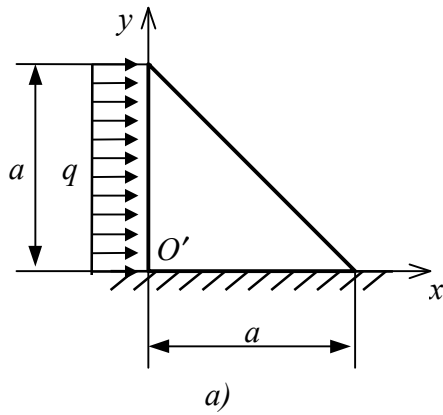


Fig. 19.3. Given solid body (a) and its single-element model (b).

$u_3, v_3$  — displacements of a free vertex

Fig. 19.4

The product  $\mathbf{A}_1 \mathbf{I}_1$  is unambiguously defined as association matrix in the relation:

$$e_1 = (\mathbf{A}_1 \mathbf{I}_1) \mathbf{t}, \quad (19.27)$$

where  $e_1$  is the strain vector of the element. In the problem under consideration the DOF-vector (19.26) includes 3-d and 6-th components of the element's nodal displacement vector (19.4). Therefore matrix  $\mathbf{A}_1 \mathbf{I}_1$  is to include 3-d and 6-th columns of the association matrix (19.13) of given element. Further evaluations are obvious:

$$\Delta_1 = a^2; \quad \mathbf{A}_1 \mathbf{I}_1 = \Delta_1^{-1} \begin{pmatrix} y_1 - y_2 & 0 \\ 0 & x_2 - x_1 \\ x_2 - x_1 & y_1 - y_2 \end{pmatrix} = a^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\mathbf{C} \mathbf{A}_1 \mathbf{I}_1 = \frac{Ea^{-1}}{1-\mu^2} \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{Ea^{-1}}{1-\mu^2} \begin{pmatrix} 0 & \mu \\ 0 & 1 \\ \frac{1-\mu}{2} & 0 \end{pmatrix};$$

$$V_1 = \frac{a^2 \delta}{2}; \quad \mathbf{R} = V_1 (\mathbf{A}_1 \mathbf{I}_1)^T \mathbf{C} (\mathbf{A}_1 \mathbf{I}_1) = \frac{a^2 \delta}{2} \cdot \frac{Ea^{-2}}{1-\mu^2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mu \\ 0 & 1 \\ \frac{1-\mu}{2} & 0 \end{pmatrix} =$$

$$= \frac{E\delta}{2(1-\mu^2)} \begin{pmatrix} \frac{1-\mu}{2} & 0 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{R}^{-1} = 2 \frac{1-\mu^2}{E\delta} \begin{pmatrix} \frac{2}{1-\mu} & 0 \\ 0 & 1 \end{pmatrix};$$

$$\mathbf{P} = \begin{pmatrix} P \\ 0 \end{pmatrix}; \quad \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \mathbf{R}^{-1} \mathbf{P} = 2 \frac{1-\mu^2}{E\delta} \begin{pmatrix} \frac{2}{1-\mu} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P \\ 0 \end{pmatrix} = 2 \frac{1-\mu^2}{E\delta} \begin{pmatrix} \frac{2}{1-\mu} P \\ 0 \end{pmatrix}.$$

Strain state:

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \mathbf{A}_1 \mathbf{I}_1 \mathbf{t} = a^{-1} \cdot 2 \frac{1-\mu^2}{E\delta} \frac{2}{1-\mu} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P \\ 0 \end{pmatrix} = \frac{4}{a\delta} \frac{1+\mu}{E} \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}.$$

Therefore, we have the deformations for single-element model:

$$\varepsilon_x = \varepsilon_y = 0; \quad \gamma_{xy} = \frac{4}{a\delta} \frac{1+\mu}{E} P.$$

The correspondent stress state is determined from the generalized Hooke's law:

$$\sigma_x = \sigma_y = 0; \quad \tau_{xy} = G\gamma_{xy} = \frac{P}{0,5a\delta}$$

Let's notice, that tangent stress is evaluated as tension-square ratio for wedge's middle horizontal section.

For self-examination we recommend to calculate SSS of construction on Fig. 19.5, using two-element model on Fig. 19.2.

## Supplement to the lecture

### Quadratic form differentiation

Let's prove, that vector of partial derivatives of quadratic form

$$y(\mathbf{x}) = \mathbf{x}^T \mathbf{B} \mathbf{x}$$

with the matrix  $\mathbf{B}$  of order  $n$ , could be written as follows:

$$\frac{\partial y}{\partial \mathbf{x}} = (\mathbf{B} + \mathbf{B}^T) \mathbf{x}. \quad (19Д.1)$$

Actually, we have with any given  $k$ :

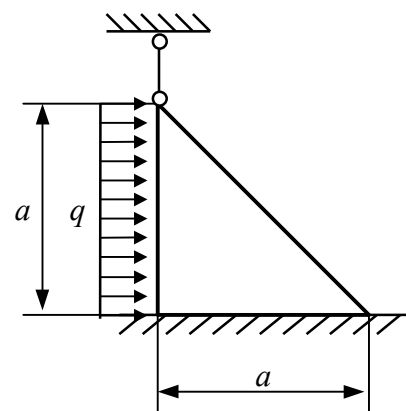


Рис. 19.5

$$y = \sum_{\substack{i \neq k \\ j \neq k}} B_{ij} x_i x_j + \sum_{j \neq k} B_{kj} x_k x_j + \sum_{i \neq k} B_{ik} x_i x_k + B_{kk} x_k^2.$$

That's why, for any  $k$ :

$$\frac{\partial y}{\partial x_k} = \sum_{j \neq k} B_{kj} x_j + \sum_{i \neq k} B_{ik} x_i + 2B_{kk} x_k = (\mathbf{B}\mathbf{x})_k + (\mathbf{B}^T \mathbf{x})_k,$$

That may be written in the form (19Д.1).

### FEM equations without simplification

Let's introduce displacement vector  $\mathbf{t}_\Sigma$  of the form (19.14), i.e. as the set of displacements of the body deprived of support. Let's introduce the vector of external forces applied to the nodes, as follows:

$$\mathbf{P}_\Sigma^T = (P_{\Sigma x1}, P_{\Sigma y1}, P_{\Sigma x2}, P_{\Sigma y2}, \dots, P_{\Sigma xn_1}, P_{\Sigma yn_1}). \quad (19Д.2)$$

Here we suppose for the nodes, which are not under loads:

$$P_{\Sigma xi} = P_{\Sigma yi} = 0. \quad (19Д.3)$$

The external forces' vector (19Д.2) differs from the load vector (19.19) by the next: for the fixed nodes the external forces should be composed of loads and bearing reactions. Suppose at first that bearings are directed along coordinate axes, i.e. restrictions for displacements of supported nodes are (19.1)—(19.3). To emphasize that the component  $P_{\Sigma i}$  specifies external force applied to fixed node along link's line, we denote it as  $X_i$ . So, for such component we can write:

$$X_i \equiv P_{\Sigma i} = P_i + R_i, \quad (19Д.4)$$

where  $P_i$  is the component of complete vector of loads (19.19),  $R_i$  is the bearing reaction.

In the case of external forces (19Д.2), applied to elastic body deprived of support, we can write for the complete potential energy instead of expression (19.22) the next expression same by sense:

$$U = \frac{1}{2} \mathbf{t}_\Sigma^T \mathbf{R} \mathbf{t}_\Sigma - \mathbf{P}_\Sigma^T \mathbf{t}_\Sigma. \quad (19Д.5)$$

Here the stiffness matrix  $\mathbf{R}$  is defined, as before, by the expression (19.18), where matrix  $\mathbf{I}_k$  specifies the conversion

$$\mathbf{t}_k = \mathbf{I}_k \mathbf{t}_\Sigma.$$

For the given external forces  $\mathbf{P}_\Sigma$  the expression (19Д.5) determines a certain function  $U = U(\mathbf{t}_\Sigma)$ . The vector  $\mathbf{t}_\Sigma$ , accordingly to the equation

$$\frac{\partial U}{\partial \mathbf{t}_\Sigma} = 0, \quad (19\text{Д.6})$$

is named stationary point of the function  $U(\mathbf{t}_\Sigma)$ . In this case the condition (19Д.6) could be transformed as follows:

$$\mathbf{R}\mathbf{t}_\Sigma - \mathbf{P}_\Sigma = 0, \quad (19\text{Д.7})$$

Unlike the equation (19.23), in the obtained equation the vector of free terms includes unknown components of type (19Д.4). But it is impossible to solve equation (19Д.7) even with the given bearing reactions, because the stiffness matrix is singular in this case. During derivation of this equation the information about body's ground attachment wasn't used; in fact, we've got vector equation for displacements of free body's points under the given external forces, which has multi-valued solution.

We shall consider equalities (19Д.7) as a system of equations for displacements and bearing reactions. If to complete this system by the conditions of grounding (19.1)—(19.3), then extended system is solved uniquely. Let's prove that.

Transpose components of displacement vector to place possible displacements (non-eliminated by links) at the beginning of sequence and then impossible displacements (eliminated by links). After this exchange it can be written:

$$\mathbf{t}_\Sigma = \begin{pmatrix} \mathbf{t} \\ \mathbf{t}_X \end{pmatrix},$$

where  $\mathbf{t}$  is sub-vector of DOFs (displacements, that isn't eliminated by links),  $\dim \mathbf{t} = n$ ;  $\mathbf{t}_X$  is sub-vector of zero displacements,  $\dim \mathbf{t}_X = 2n_1 - n$ . Let's do the same transposition for components of external forces' vector and represent one as follows:

$$\mathbf{P}_\Sigma = \begin{pmatrix} \mathbf{P} \\ \mathbf{X} \end{pmatrix},$$

where  $\mathbf{P}$  is sub-vector of loads directed along the possible displacements;  $\mathbf{X}$  is sub-vector of unknown external forces (19Д.4), i.e. forces inclusive reactions. Further we'll consider vectors  $\mathbf{t}_\Sigma$  and  $\mathbf{P}_\Sigma$  with transposed components. The expression for potential energy (19Д.5) remains true (if we use the matrices  $\mathbf{I}_k$  with transposed columns for calculation of the matrix  $\mathbf{R}$ ). Stationary condition (19Д.7) do not alter as well. Let's represent the stiffness matrix with the boxes:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{pmatrix}, \quad (19\text{Д.8})$$

where  $\mathbf{R}_{11}$  is the box of order  $n$ . The equation (19Д.7) together with conditions of grounding can be expressed as follows:

$$\left\{ \begin{array}{l} \left( \begin{array}{cc} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{array} \right) \left( \begin{array}{c} \mathbf{t} \\ \mathbf{t}_X \end{array} \right) - \left( \begin{array}{c} \mathbf{P} \\ \mathbf{X} \end{array} \right) = 0; \\ \mathbf{t}_X = 0. \end{array} \right. \quad (19Д.9)$$

Let's substitute the second equation of this system into the first one:

$$\left\{ \begin{array}{l} \mathbf{R}_{11}\mathbf{t} - \mathbf{P} = 0; \\ \mathbf{R}_{21}\mathbf{t} - \mathbf{X} = 0. \end{array} \right. \quad (19Д.10)$$

Vectors  $\mathbf{t}$  and  $\mathbf{P}$  coincide with corresponding vectors of simplified FEM, considered in the lecture. It's also easily established, that box  $\mathbf{R}_{11}$  coincides with stiffness matrix  $\mathbf{R}$  of simplified FEM. It means the first equation of the system (19Д.10) coincides with equation (19.23) of simplified FEM. Let's establish, that this equation is solved uniquely.

We have to prove that  $\mathbf{R}_{11}$  is nonsingular matrix. The proof is based on the next statements:

Stationary conditions (19Д.6) together with linkage conditions  $\mathbf{t}_X = 0$  determine system displacements in equilibrium;

There are displacements in the stable system only under the loads which not coincide with bearing reactions by an application point and direction.

The first statement means there is state of body equilibrium under the condition (19Д.10). Loads in the second statement compose vector  $\mathbf{P}$ . The statement means it is impossible that  $\mathbf{R}_{11}\mathbf{t} = 0$  for  $\mathbf{t} \neq 0$ . Hence the matrix  $\mathbf{R}_{11}$  is nonsingular [11]. So the first of equations (19Д.10) is solved uniquely of  $\mathbf{t}$ . The second equation is solved after the first one concerning to vector of external forces  $\mathbf{X}$ , and gives bearing reactions.

Now, let's demonstrate the way of calculation oblique links' reactions. These links causes additional condition for displacements:

$$\mathbf{B}^T \mathbf{t}_\Sigma = 0, \quad (19Д.11)$$

where  $\mathbf{B}$  is matrix of additional conditions for nodal displacements, having dimensionality  $2n_1 \times k$  ( $k$  is the number of scalar conditions). For example, if there is one bearing defined by condition  $u = kv$ , than the matrix  $\mathbf{B}$  is the vector-column having only two non-zero components of value 1 and  $-k$ .

Let's remove all the bearings eliminating DOFs, and retain only oblique bearings. For the such system the complete potential energy  $U = U(\mathbf{t}_\Sigma)$  takes the form (19Д.5) as before. The minimum point of function  $U(\mathbf{t}_\Sigma)$  corresponds to stable equilibrium state under the condition (19Д.11). This point we determine with Lagrange multiplier method. Compose Lagrange function as follows:

$$L = \frac{1}{2} \mathbf{t}_\Sigma^T \mathbf{R} \mathbf{t}_\Sigma - \mathbf{P}_\Sigma^T \mathbf{t}_\Sigma - \mathbf{l}^T \mathbf{B}^T \mathbf{t}_\Sigma, \quad (19\text{Д.}12)$$

where  $\mathbf{l}$  is the vector of Lagrange multipliers. Deriving this function by  $\mathbf{t}_\Sigma$ , we obtain the equilibrium conditions:

$$\begin{cases} \mathbf{R} \mathbf{t}_\Sigma - \mathbf{P}_\Sigma - \mathbf{B} \mathbf{l} = 0; \\ \mathbf{B}^T \mathbf{t}_\Sigma = 0. \end{cases} \quad (19\text{Д.}13)$$

Here the unknown variables are vectors  $\mathbf{t}_\Sigma$ ,  $\mathbf{l}$ , and the components of vector  $\mathbf{P}_\Sigma$  of the form (19Д.4).

Let exist the eliminated DOFs, i.e.  $n < 2n_1$ . Then the system (19Д.13) has to be solved under the condition  $\mathbf{t}_X = 0$ . Using partition form of stiffness matrix, we get the system of equations:

$$\begin{cases} \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{t}_X \end{pmatrix} - \begin{pmatrix} \mathbf{P} \\ \mathbf{X} \end{pmatrix} - \mathbf{B} \mathbf{l} = 0; \\ \mathbf{t}_X = 0; \\ \mathbf{B}^T \begin{pmatrix} \mathbf{t} \\ \mathbf{t}_X \end{pmatrix} = 0. \end{cases} \quad (19\text{Д.}14)$$

This system is close to the system (19Д.9) by its form, and the way of solution is similar for both systems: at first there determined unknown  $\mathbf{t}$  and  $\mathbf{l}$ , and then vector  $\mathbf{X}$  inclusive bearing reactions. Note, that it's impossible to obtain reactions of oblique bearings but only those, which have eliminated DOFs  $x$  or  $y$ .

## REFERENCES

11. Iljin V.A., Pozniak E.G. Linear algebra (Linejnaja algebra). "Nauka", Moscow, 1978.
16. Rzhantsin A.R. Structural Mechanics (Stroitel'naja mehanika). "Visshaia shkola", Moscow, 1982.
30. ANSYS, Inc. Theory. Release 5.7. Edited by Peter Kohnke. ANSYS, Inc., Canonsburg, 2001.
31. Rusakov A.I. Rational Algorithm of Finite Element Method. In Coll.: Proceedings of All-Russian Scientific Conference "Transport-2006". Part 3. Rostov-on-Don: Rostov State University of Transport Communication, 2006, pp. 87-89. (in Russian).